LESSON NOTE

Row Echelon Form *and* Reduced Row Echelon Form

Matrices || Self study notes with worked examples and practice

By the end of this note you will be able to

1. Use the three row operations and write them in the correct notation.
2. Say whether a matrix is in row echelon form (REF) and reduce a matrix to REF.
3. Say whether a matrix is in reduced row echelon form (RREF) and reduce a matrix to RREF.
4. Use REF and RREF to solve simultaneous equations.

1 Quick recap: rows, columns and the augmented matrix

A **matrix** is a rectangle of numbers. **Rows** go across and **columns** go down. We label rows R_1, R_2, R_3 from the top.

The **leading entry** of a row is the first number in that row (reading from the left) that is not zero. A row made up only of zeros is called a **zero row** and has no leading entry.

When we use matrices to solve simultaneous equations, we write only the numbers. This is called the **augmented matrix**. The vertical line stands for the equals sign.

$$\begin{aligned} 2x + y - z &= 3 \\ x - 3y + 2z &= 4 \\ 3x + 2y + z &= 10 \end{aligned} \quad \textbf{becomes} \quad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & -3 & 2 & 4 \\ 3 & 2 & 1 & 10 \end{array}\right]$$

Watch out: a missing letter means its number is 0, and a letter on its own means its number is 1. For example, $x + z = 4$ gives the row 1 0 1 | 4.

2 The three row operations

These are the only three moves you are allowed to make. None of them changes the answers to the equations, which is why they are safe to use.

Operation	What it means	How we write it
1. Swap	Two rows change places.	$R_1 \leftrightarrow R_2$
2. Multiply or divide	Every number in a row is multiplied (or divided) by the same nonzero number.	$R_2 \to R_2 \div 3$

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3. Add a multiple	Add (or take away) a multiple of another row.	$R_3 \to R_3 - 2R_1$

How to read $R_3 \to R_3 - 2R_1$: "the new Row 3 is the old Row 3 take away 2 lots of Row 1". Work it out one column at a time. Only Row 3 changes; Row 1 stays exactly as it is.

3 Concept 1: Row echelon form (REF)

"Echelon" means a staircase. A matrix is in **row echelon form** when it follows these three rules:

Rule 1 Any zero rows are at the **bottom**.

Rule 2 The leading entry of every nonzero row is **1** (we call it a **leading 1**).

Rule 3 Each leading 1 is to the **right** of the leading 1 in the row above. So everything **below** a leading 1 is zero.

$$\begin{bmatrix} \textcircled{1} & 4 & 2 & 7 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The circled numbers are the leading 1s. Notice how they step down and to the right like a staircase, with only zeros underneath them and the zero row at the bottom.

Good to know: some textbooks only ask for the leading entries to be nonzero instead of 1. In these notes we always make them 1, because this makes the next stage (RREF) easier.

How to reduce a matrix to REF

1. **Look at Column 1.** If the top number is 0, swap with a row below that has a nonzero number there. (A row starting with 1 is the best choice.)
2. **Make the top number 1** by dividing the whole row by it.
3. **Clear below it.** Use "add a multiple" to turn every number underneath that leading 1 into 0.
4. **Move down one row and right one column** and repeat steps 1 to 3, ignoring the rows already finished.
5. **Stop** when the staircase is complete.

Choosing the multiple: to clear a number k under a leading 1, use $R \to R - k \times$ (leading row). If k is negative, taking away a negative becomes adding, for example $R_3 \to R_3 + 2R_1$ clears a -2.

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Worked examples

Worked Example 1

[4 marks]

Which of these matrices are in row echelon form? Give a reason for each.

$$\textbf{A}\begin{bmatrix}1&3&2\\0&1&4\\0&0&1\end{bmatrix}\quad \textbf{B}\begin{bmatrix}1&2\\0&0\\0&1\end{bmatrix}\quad \textbf{C}\begin{bmatrix}1&4&5\\0&0&1\\0&1&3\end{bmatrix}\quad \textbf{D}\begin{bmatrix}1&-2&3\\0&1&5\\0&0&0\end{bmatrix}$$

Step 1: Check matrix A

$$\begin{bmatrix}\textbf{1}&3&2\\0&\textbf{1}&4\\0&0&\textbf{1}\end{bmatrix}$$

Every leading entry is 1, each one sits to the right of the one above, and there are no zero rows.
Yes, A is in row echelon form.

Step 2: Check matrix B

$$\begin{bmatrix}\textbf{1}&2\\0&0\\0&\textbf{1}\end{bmatrix}$$

Row 2 is a zero row but it is **not** at the bottom. Row 3 is below it and has numbers in it.
No, B is not in row echelon form.

Step 3: Check matrix C

$$\begin{bmatrix}\textbf{1}&4&5\\0&0&\textbf{1}\\0&\textbf{1}&3\end{bmatrix}$$

The leading 1 of Row 3 is in Column 2, which is to the **left** of the leading 1 of Row 2 (Column 3). The staircase goes backwards.
No, C is not in row echelon form.

Step 4: Check matrix D

$$\begin{bmatrix}\textbf{1}&-2&3\\0&\textbf{1}&5\\0&0&0\end{bmatrix}$$

Leading 1s step to the right, and the zero row is at the bottom.
Yes, D is in row echelon form.

ANSWER In row echelon form: **A, D** Not in row echelon form: **B, C**

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Worked Example 2

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$$

Step 1: Make the leading entry of Row 1 equal to 1

$R_1 \rightarrow R_1 \div 2$

$2 \div 2$	$4 \div 2$
$= 1$	$= 2$

NEW MATRIX $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 3R_1$

$3 - 3(1)$	$7 - 3(2)$
$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textcircled{1} & 2 \\ 0 & \textcircled{1} \end{bmatrix}$

Worked Example 3

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 14 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$8 - 2(3)$	$14 - 2(5)$
$= 0$	$= 2$	$= 4$

NEW MATRIX $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix}$

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Step 2: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div 2$

$0 \div 2$	$2 \div 2$	$4 \div 2$
$= 0$	$= 1$	$= 2$

NEW MATRIX $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$

ANSWER $\begin{bmatrix} \mathbf{1} & 3 & 5 \\ 0 & \mathbf{1} & 2 \end{bmatrix}$

Worked Example 4

[4 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 3 \\ 2 & 4 & 10 \end{bmatrix}$$

Step 1: Swap rows to put a 1 at the top of Column 1

$R_1 \leftrightarrow R_2$

Row 1 and Row 2 change places.

NEW MATRIX $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 2 & 4 & 10 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_3 \rightarrow R_3 - 2R_1$

$2 - 2(1)$	$4 - 2(1)$	$10 - 2(3)$
$= 0$	$= 2$	$= 4$

NEW MATRIX $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix}$

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Step 3: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div 2$

$0 \div 2$	$2 \div 2$	$4 \div 2$
$= 0$	$= 1$	$= 2$

NEW MATRIX

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

Step 4: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 - 2R_2$

$0 - 2(0)$	$2 - 2(1)$	$4 - 2(2)$
$= 0$	$= 0$	$= 0$

NEW MATRIX

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

ANSWER

$$\begin{bmatrix} \textbf{1} & 1 & 3 \\ 0 & \textbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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Worked Example 5

[4 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ -2 & -3 & 5 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 3R_1$

$3 - 3(1)$	$7 - 3(2)$	$-2 - 3(-1)$
$= 0$	$= 1$	$= 1$

$R_3 \rightarrow R_3 + 2R_1$

$-2 + 2(1)$	$-3 + 2(2)$	$5 + 2(-1)$
$= 0$	$= 1$	$= 3$

NEW MATRIX $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 - R_2$

$0 - (0)$	$1 - (1)$	$3 - (1)$
$= 0$	$= 0$	$= 2$

NEW MATRIX $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Step 3: Make the leading entry of Row 3 equal to 1

$R_3 \rightarrow R_3 \div 2$

$0 \div 2$	$0 \div 2$	$2 \div 2$
$= 0$	$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

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ANSWER

$$\begin{bmatrix} \textbf{1} & 2 & -1 \\ 0 & \textbf{1} & 1 \\ 0 & 0 & \textbf{1} \end{bmatrix}$$

Worked Example 6

[6 marks]

Use row echelon form to solve the simultaneous equations.

$x + y + z = 6$

$2x + 3y + z = 11$

$x - y + 2z = 5$

Step 1: Write the augmented matrix

Put the numbers in front of x, y, z on the left of the line and the answers on the right.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 11 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Step 2: Make every entry below the leading 1 in Column 1 zero

$\mathbf{R_2 \rightarrow R_2 - 2R_1}$

$2 - 2(1)$	$3 - 2(1)$	$1 - 2(1)$	$11 - 2(6)$
= 0	**= 1**	**= −1**	**= −1**

$\mathbf{R_3 \rightarrow R_3 - R_1}$

$1 - (1)$	$-1 - (1)$	$2 - (1)$	$5 - (6)$
= 0	**= −2**	**= 1**	**= −1**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

Step 3: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 + 2R_2$

$0 + 2(0)$	$-2 + 2(1)$	$1 + 2(-1)$	$-1 + 2(-1)$
$= 0$	$= 0$	$= -1$	$= -3$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Step 4: Make the leading entry of Row 3 equal to 1

$R_3 \rightarrow R_3 \div (-1)$

$0 \div (-1)$	$0 \div (-1)$	$-1 \div (-1)$	$-3 \div (-1)$
$= 0$	$= 0$	$= 1$	$= 3$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 5: Back substitution

Turn each row back into an equation and work from the bottom row upwards.

Row 3 already gives the value directly.

$\boldsymbol{z = 3}$

$y - z = -1$

$y - (3) = -1$

$y - 3 = -1$

$y = -1 + 3$ *(carry 3 over, its sign changes)*

$\boldsymbol{y = 2}$

$x + y + z = 6$

$x + (2) + (3) = 6$

$x + 5 = 6$

$x = 6 - 5$ *(carry 5 over, its sign changes)*

$\boldsymbol{x = 1}$

ANSWER

$$\left[\begin{array}{ccc|c} \textbf{1} & 1 & 1 & 6 \\ 0 & \textbf{1} & -1 & -1 \\ 0 & 0 & \textbf{1} & 3 \end{array}\right]$$

$\boldsymbol{x = 1,\ y = 2,\ z = 3}$

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4 Concept 2: Reduced row echelon form (RREF)

RREF is REF with one extra clean up rule. A matrix is in **reduced row echelon form** when:

Rules 1 to 3 It is already in row echelon form.

Rule 4 Every leading 1 is the **only** nonzero number in its column. So there are zeros **above** each leading 1 as well as below it.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 5 \\ 0 & 0 & \textcircled{1} & 2 \end{array}\right]$$

Each leading 1 sits alone in its column. Reading the rows gives the answers straight away: $x = 3$, $y = 5$, $z = 2$. This is why RREF is so useful.

How to reduce a matrix to RREF

1. **First reach REF** using the method from Concept 1.
2. **Start with the last leading 1** (the one furthest down and right).
3. **Clear above it.** Use "add a multiple" to make every number above that leading 1 equal to 0.
4. **Move up to the next leading 1** and clear above it too. Keep going until you reach the top.

Why work upwards from the bottom? The bottom rows have the most zeros, so using them to clear the rows above does not undo any zeros you have already made.

REF or RREF? A matrix can have many different REFs, depending on the steps you choose. But every matrix has exactly **one** RREF. If your RREF matches the answer, your working was correct.

Worked examples

Worked Example 7

[4 marks]

Which of these matrices are in reduced row echelon form? Give a reason for each.

$$\mathbf{P}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \mathbf{Q}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad \mathbf{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{S}\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 1: Check matrix P

$$\begin{bmatrix} \textcircled{1} & 0 & 3 \\ 0 & \textcircled{1} & 2 \end{bmatrix}$$

Leading 1s step to the right, and each leading 1 is the only nonzero number in its column. The 3 and 2 are in a column with no leading 1, so they are allowed.
Yes, P is in reduced row echelon form.

Step 2: Check matrix Q

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & 4 \end{bmatrix}$$

It is in row echelon form, but the leading 1 of Row 2 (Column 2) has a 2 **above** it.
No, Q is not in reduced row echelon form.

Step 3: Check matrix R

$$\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

This is the identity matrix. Every leading 1 has zeros above and below it.
Yes, R is in reduced row echelon form.

Step 4: Check matrix S

$$\begin{bmatrix} \textcircled{1} & 0 & 5 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}$$

The leading 1 of Row 2 is in Column 3, and there is a 5 above it in Row 1.
No, S is not in reduced row echelon form.

ANSWER In reduced row echelon form: **P, R** Not in reduced row echelon form: **Q, S**

Worked Example 8

[3 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$5 - 2(3)$
$= 0$	$= -1$

NEW MATRIX $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$

Step 2: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div (-1)$

$0 \div (-1)$	$-1 \div (-1)$
$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Step 3: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 - 3R_2$

$1 - 3(0)$	$3 - 3(1)$
$= 1$	$= 0$

NEW MATRIX $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Worked Example 9

[5 marks]

Use reduced row echelon form to solve the simultaneous equations.

$2x + y = 7$

$x - y = -1$

Step 1: Write the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & -1 & -1 \end{array}\right]$$

Step 2: Swap rows to put a 1 at the top of Column 1

$R_1 \leftrightarrow R_2$

Row 1 and Row 2 change places.

NEW MATRIX $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 1 & 7 \end{array}\right]$

Step 3: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$1 - 2(-1)$	$7 - 2(-1)$
$= 0$	$= 3$	$= 9$

NEW MATRIX $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 9 \end{array}\right]$

Step 4: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div 3$

$0 \div 3$	$3 \div 3$	$9 \div 3$
$= 0$	$= 1$	$= 3$

NEW MATRIX $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array}\right]$

Step 5: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 + R_2$

$1 + (0)$	$-1 + (1)$	$-1 + (3)$
$= 1$	$= 0$	$= 2$

NEW MATRIX $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$

Step 6: Read off the answers

Each row now says one variable equals one number.

$x = 2$

$y = 3$

ANSWER $\left[\begin{array}{cc|c} \textbf{1} & 0 & 2 \\ 0 & \textbf{1} & 3 \end{array}\right]$ $x = 2, \ y = 3$

Worked Example 10

[5 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_3 \rightarrow R_3 - 5R_1$

$5 - 5(1)$	$6 - 5(2)$	$0 - 5(3)$
$= 0$	$= -4$	$= -15$

NEW MATRIX $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -4 & -15 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 + 4R_2$

$0 + 4(0)$	$-4 + 4(1)$	$-15 + 4(4)$
$= 0$	$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Step 3: Make every entry above the leading 1 in Column 3 zero

$R_1 \rightarrow R_1 - 3R_3$

$1 - 3(0)$	$2 - 3(0)$	$3 - 3(1)$
$= 1$	$= 2$	$= 0$

$R_2 \rightarrow R_2 - 4R_3$

$0 - 4(0)$	$1 - 4(0)$	$4 - 4(1)$
$= 0$	$= 1$	$= 0$

NEW MATRIX $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Step 4: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 - 2R_2$

$1 - 2(0)$	$2 - 2(1)$	$0 - 2(0)$
$= 1$	$= 0$	$= 0$

NEW MATRIX $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textbf{1} & 0 & 0 \\ 0 & \textbf{1} & 0 \\ 0 & 0 & \textbf{1} \end{bmatrix}$

Worked Example 11

[6 marks]

Use reduced row echelon form to solve the simultaneous equations.

$$x + 2y + z = 3$$
$$2x + y - z = 0$$
$$x - y + 2z = 9$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 0 \\ 1 & -1 & 2 & 9 \end{array}\right]$$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$1 - 2(2)$	$-1 - 2(1)$	$0 - 2(3)$
$= 0$	$= -3$	$= -3$	$= -6$

$R_3 \rightarrow R_3 - R_1$

$1 - (1)$	$-1 - (2)$	$2 - (1)$	$9 - (3)$
$= 0$	$= -3$	$= 1$	$= 6$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & -3 & 1 & 6 \end{array}\right]$

Step 3: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div (-3)$

$0 \div (-3)$	$-3 \div (-3)$	$-3 \div (-3)$	$-6 \div (-3)$
= 0	**= 1**	**= 1**	**= 2**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & 1 & 6 \end{array}\right]$$

Step 4: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 + 3R_2$

$0 + 3(0)$	$-3 + 3(1)$	$1 + 3(1)$	$6 + 3(2)$
= 0	**= 0**	**= 4**	**= 12**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

Step 5: Make the leading entry of Row 3 equal to 1

$R_3 \rightarrow R_3 \div 4$

$0 \div 4$	$0 \div 4$	$4 \div 4$	$12 \div 4$
= 0	**= 0**	**= 1**	**= 3**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 6: Make every entry above the leading 1 in Column 3 zero

$R_1 \rightarrow R_1 - R_3$

$1 - (0)$	$2 - (0)$	$1 - (1)$	$3 - (3)$
= 1	**= 2**	**= 0**	**= 0**

$R_2 \rightarrow R_2 - R_3$

$0 - (0)$	$1 - (0)$	$1 - (1)$	$2 - (3)$
= 0	**= 1**	**= 0**	**= −1**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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Step 7: Make every entry above the leading 1 in Column 2 zero

$R_1 \to R_1 - 2R_2$

$1-2(0)$	$2-2(1)$	$0-2(0)$	$0-2(-1)$
$=1$	$=0$	$=0$	$=2$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 8: Read off the answers

Each row now says one variable equals one number.

$x = 2$

$y = -1$

$z = 3$

ANSWER

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 2 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & \textcircled{1} & 3 \end{array}\right]$$

$x = 2,\ y = -1,\ z = 3$

Worked Example 12

[6 marks]

Use reduced row echelon form to solve the simultaneous equations.

$2x + 2y + z = 2$

$4x - y + 2z = -1$

$2x + 3y - z = 5$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 4 & -1 & 2 & -1 \\ 2 & 3 & -1 & 5 \end{array}\right]$$

Step 2: Make the leading entry of Row 1 equal to 1

$R_1 \to R_1 \div 2$

$2 \div 2$	$2 \div 2$	$1 \div 2$	$2 \div 2$
$= 1$	$= 1$	$= \frac{1}{2}$	$= 1$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 1 \\ 4 & -1 & 2 & -1 \\ 2 & 3 & -1 & 5 \end{array}\right]$$

Step 3: Make every entry below the leading 1 in Column 1 zero

$R_2 \to R_2 - 4R_1$

$4 - 4(1)$	$-1 - 4(1)$	$2 - 4(\frac{1}{2})$	$-1 - 4(1)$
$= 0$	$= -5$	$= 0$	$= -5$

$R_3 \to R_3 - 2R_1$

$2 - 2(1)$	$3 - 2(1)$	$-1 - 2(\frac{1}{2})$	$5 - 2(1)$
$= 0$	$= 1$	$= -2$	$= 3$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 1 \\ 0 & -5 & 0 & -5 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

Step 4: Swap rows to put a 1 at the top of Column 2

$R_2 \leftrightarrow R_3$

Row 2 and Row 3 change places.

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & 3 \\ 0 & -5 & 0 & -5 \end{array}\right]$$

Step 5: Make every entry below the leading 1 in Column 2 zero

$R_3 \to R_3 + 5R_2$

$0 + 5(0)$	$-5 + 5(1)$	$0 + 5(-2)$	$-5 + 5(3)$
$= 0$	$= 0$	$= -10$	$= 10$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -10 & 10 \end{array}\right]$$

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Step 6: Make the leading entry of Row 3 equal to 1

$R_3 \to R_3 \div (-10)$

$0 \div (-10)$	$0 \div (-10)$	$-10 \div (-10)$	$10 \div (-10)$
$= 0$	$= 0$	$= 1$	$= -1$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Step 7: Make every entry above the leading 1 in Column 3 zero

$R_1 \to R_1 - \frac{1}{2}R_3$

$1 - \frac{1}{2}(0)$	$1 - \frac{1}{2}(0)$	$\frac{1}{2} - \frac{1}{2}(1)$	$1 - \frac{1}{2}(-1)$
$= 1$	$= 1$	$= 0$	$= \frac{3}{2}$

$R_2 \to R_2 + 2R_3$

$0 + 2(0)$	$1 + 2(0)$	$-2 + 2(1)$	$3 + 2(-1)$
$= 0$	$= 1$	$= 0$	$= 1$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Step 8: Make every entry above the leading 1 in Column 2 zero

$R_1 \to R_1 - R_2$

$1 - (0)$	$1 - (1)$	$0 - (0)$	$\frac{3}{2} - (1)$
$= 1$	$= 0$	$= 0$	$= \frac{1}{2}$

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Step 9: Read off the answers

Each row now says one variable equals one number.

$$x = \frac{1}{2}$$
$$y = 1$$
$$z = -1$$

ANSWER

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & \frac{1}{2} \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & \textcircled{1} & -1 \end{array}\right] \qquad x = \frac{1}{2},\ y = 1,\ z = -1$$

What if a whole row becomes zeros? When solving equations, a row such as 0 0 0 | 0 means one equation was a copy of the others, so there are **infinitely many** solutions. A row such as 0 0 0 | 5 says 0 = 5, which is impossible, so there is **no solution**.

5 Key points to remember

	Row echelon form (REF)	Reduced row echelon form (RREF)
Zero rows	At the bottom	At the bottom
Leading entries	Each is 1, stepping to the right	Each is 1, stepping to the right
Below each leading 1	All zeros	All zeros
Above each leading 1	Anything	All zeros
Is it unique?	No, it depends on your steps	Yes, always the same
Solving equations	Needs back substitution	Read the answers straight off

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6 Practice questions

Try these on paper before looking at the answers. Show every row operation you use.

Concept 1: Row echelon form

Question 1

[3 marks]

Which of these matrices are in row echelon form? Give a reason for each.

E $\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ **F** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **G** $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 4 & 2 \\ 0 & 1 & 6 \end{bmatrix}$

Question 2

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$$

Question 3

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \end{bmatrix}$$

Question 4

[4 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 5 & 9 \end{bmatrix}$$

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Question 5

[6 marks]

Use row echelon form and back substitution to solve

$$x + 2y + z = 7$$
$$2x + 5y + 3z = 18$$
$$x + 3y + 3z = 13$$

Concept 2: Reduced row echelon form

Question 6

[3 marks]

Which of these matrices are in reduced row echelon form? Give a reason for each.

H $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **J** $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ **K** $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 7

[3 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

Question 8

[5 marks]

Use reduced row echelon form to solve

$$x + 3y = 5$$
$$2x + 7y = 12$$

Question 9

[5 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

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Question 10

[6 marks]

Use reduced row echelon form to solve

$$x + y + z = 2$$
$$x + 2y + 3z = 2$$
$$2x + y - z = 3$$

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7 Answer key

Answer summary

Question	Final answer	Marks
1	In row echelon form: **E** Not in row echelon form: **F, G**	3
2	$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	3
3	$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$	3
4	$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$	4
5	$x = 1,\ y = 2,\ z = 2$	6
6	In reduced row echelon form: **H, J** Not in reduced row echelon form: **K**	3
7	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	3
8	$x = -1,\ y = 2$	5
9	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	5
10	$x = 3,\ y = -2,\ z = 1$	6

Note for Questions 2, 3 and 4: REF is not unique, so a correct answer may look different if you used different steps. Check that your matrix follows all three REF rules.

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Full worked solutions

Question 1

[3 marks]

Which of these matrices are in row echelon form? Give a reason for each.

$$\mathbf{E}\begin{bmatrix}1&5&2\\0&1&3\\0&0&0\end{bmatrix}\quad \mathbf{F}\begin{bmatrix}1&2&3\\0&3&1\\0&0&1\end{bmatrix}\quad \mathbf{G}\begin{bmatrix}0&0&0\\1&4&2\\0&1&6\end{bmatrix}$$

Step 1: Check matrix E

$$\begin{bmatrix}\textcircled{1}&5&2\\0&\textcircled{1}&3\\0&0&0\end{bmatrix}$$

Leading 1s step to the right and the zero row is at the bottom.
Yes, E is in row echelon form.

Step 2: Check matrix F

$$\begin{bmatrix}\textcircled{1}&2&3\\0&\textcircled{3}&1\\0&0&\textcircled{1}\end{bmatrix}$$

The leading entry of Row 2 is 3, not 1.
No, F is not in row echelon form.

Step 3: Check matrix G

$$\begin{bmatrix}0&0&0\\\textcircled{1}&4&2\\0&\textcircled{1}&6\end{bmatrix}$$

The zero row is at the top instead of the bottom.
No, G is not in row echelon form.

ANSWER In row echelon form: **E** Not in row echelon form: **F, G**

Question 2

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$$

Step 1: Make the leading entry of Row 1 equal to 1

$R_1 \rightarrow R_1 \div 3$

$3 \div 3$	$6 \div 3$
= 1	**= 2**

NEW MATRIX $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$5 - 2(2)$
= 0	**= 1**

NEW MATRIX $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textbf{1} & 2 \\ 0 & \textbf{1} \end{bmatrix}$

Question 3

[3 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 3R_1$

$3 - 3(1)$	$8 - 3(2)$	$14 - 3(4)$
= 0	**= 2**	**= 2**

NEW MATRIX $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

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Step 2: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div 2$

$0 \div 2$	$2 \div 2$	$2 \div 2$
$= 0$	$= 1$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textbf{1} & 2 & 4 \\ 0 & \textbf{1} & 1 \end{bmatrix}$

Question 4

[4 marks]

Reduce this matrix to row echelon form.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 5 & 9 \end{bmatrix}$$

Step 1: Swap rows to put a 1 at the top of Column 1

$R_1 \leftrightarrow R_2$

Row 1 and Row 2 change places.

NEW MATRIX $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 2 & 5 & 9 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_3 \rightarrow R_3 - 2R_1$

$2 - 2(1)$	$5 - 2(3)$	$9 - 2(4)$
$= 0$	$= -1$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

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Step 3: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 + R_2$

$0 + (0)$	$-1 + (1)$	$1 + (2)$
$= 0$	$= 0$	$= 3$

NEW MATRIX $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Step 4: Make the leading entry of Row 3 equal to 1

$R_3 \rightarrow R_3 \div 3$

$0 \div 3$	$0 \div 3$	$3 \div 3$
$= 0$	$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textcircled{1} & 3 & 4 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$

Question 5

[6 marks]

Use row echelon form and back substitution to solve

$x + 2y + z = 7$

$2x + 5y + 3z = 18$

$x + 3y + 3z = 13$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 5 & 3 & 18 \\ 1 & 3 & 3 & 13 \end{array}\right]$$

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Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$5 - 2(2)$	$3 - 2(1)$	$18 - 2(7)$
= 0	**= 1**	**= 1**	**= 4**

$R_3 \rightarrow R_3 - R_1$

$1 - (1)$	$3 - (2)$	$3 - (1)$	$13 - (7)$
= 0	**= 1**	**= 2**	**= 6**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 6 \end{array}\right]$$

Step 3: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 - R_2$

$0 - (0)$	$1 - (1)$	$2 - (1)$	$6 - (4)$
= 0	**= 0**	**= 1**	**= 2**

NEW MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 4: Back substitution

Turn each row back into an equation and work from the bottom row upwards.

Row 3 already gives the value directly.

$\boldsymbol{z = 2}$

$y + z = 4$

$y + (2) = 4$

$y + 2 = 4$

$y = 4 - 2$ *(carry 2 over, its sign changes)*

$\boldsymbol{y = 2}$

$x + 2y + z = 7$

$x + 2(2) + (2) = 7$

$x + 6 = 7$

$x = 7 - 6$ *(carry 6 over, its sign changes)*

$\boldsymbol{x = 1}$

ANSWER

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 7 \\ 0 & \textcircled{1} & 1 & 4 \\ 0 & 0 & \textcircled{1} & 2 \end{array}\right] \quad x = 1,\ y = 2,\ z = 2$$

Question 6

[3 marks]

Which of these matrices are in reduced row echelon form? Give a reason for each.

$$\mathbf{H}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{J}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix} \quad \mathbf{K}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 1: Check matrix H

$$\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Each leading 1 has zeros above and below it, and the zero row is at the bottom.
Yes, H is in reduced row echelon form.

Step 2: Check matrix J

$$\begin{bmatrix} \textcircled{1} & 0 & 4 \\ 0 & \textcircled{1} & 7 \end{bmatrix}$$

Each leading 1 is alone in its column. The last column has no leading 1, so 4 and 7 are allowed.
Yes, J is in reduced row echelon form.

Step 3: Check matrix K

$$\begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

The leading 1 of Row 2 (Column 2) has a 3 above it.
No, K is not in reduced row echelon form.

ANSWER In reduced row echelon form: **H, J** Not in reduced row echelon form: **K**

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Question 7

[3 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

Step 1: Swap rows to put a 1 at the top of Column 1

$R_1 \leftrightarrow R_2$

Row 1 and Row 2 change places.

NEW MATRIX $\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$6 - 2(4)$
$= 0$	$= -2$

NEW MATRIX $\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}$

Step 3: Make the leading entry of Row 2 equal to 1

$R_2 \rightarrow R_2 \div (-2)$

$0 \div (-2)$	$-2 \div (-2)$
$= 0$	$= 1$

NEW MATRIX $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Step 4: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 - 4R_2$

$1 - 4(0)$	$4 - 4(1)$
$= 1$	$= 0$

NEW MATRIX $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Question 8

[5 marks]

Use reduced row echelon form to solve

$$x + 3y = 5$$
$$2x + 7y = 12$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 7 & 12 \end{array}\right]$$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$7 - 2(3)$	$12 - 2(5)$
= 0	**= 1**	**= 2**

NEW MATRIX $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 2 \end{array}\right]$

Step 3: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 - 3R_2$

$1 - 3(0)$	$3 - 3(1)$	$5 - 3(2)$
= 1	**= 0**	**= −1**

NEW MATRIX $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$

Step 4: Read off the answers

Each row now says one variable equals one number.

$$x = -1$$
$$y = 2$$

ANSWER $\left[\begin{array}{cc|c} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 2 \end{array}\right]$ $x = -1,\ y = 2$

Question 9

[5 marks]

Reduce this matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

Step 1: Make every entry below the leading 1 in Column 1 zero

$R_2 \rightarrow R_2 - 2R_1$

$2 - 2(1)$	$3 - 2(1)$	$5 - 2(2)$
= 0	**= 1**	**= 1**

$R_3 \rightarrow R_3 - R_1$

$1 - (1)$	$2 - (1)$	$4 - (2)$
= 0	**= 1**	**= 2**

NEW MATRIX $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Step 2: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 - R_2$

$0 - (0)$	$1 - (1)$	$2 - (1)$
= 0	**= 0**	**= 1**

NEW MATRIX $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Step 3: Make every entry above the leading 1 in Column 3 zero

$R_1 \rightarrow R_1 - 2R_3$

$1 - 2(0)$	$1 - 2(0)$	$2 - 2(1)$
= 1	**= 1**	**= 0**

$R_2 \rightarrow R_2 - R_3$

$0 - (0)$	$1 - (0)$	$1 - (1)$
= 0	**= 1**	**= 0**

NEW MATRIX $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Step 4: Make every entry above the leading 1 in Column 2 zero

$R_1 \to R_1 - R_2$

$1-(0)$	$1-(1)$	$0-(0)$
$=1$	$=0$	$=0$

NEW MATRIX $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ANSWER $\begin{bmatrix} \textbf{1} & 0 & 0 \\ 0 & \textbf{1} & 0 \\ 0 & 0 & \textbf{1} \end{bmatrix}$

Question 10

[6 marks]

Use reduced row echelon form to solve

$$x + y + z = 2$$
$$x + 2y + 3z = 2$$
$$2x + y - z = 3$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

Step 2: Make every entry below the leading 1 in Column 1 zero

$R_2 \to R_2 - R_1$

$1-(1)$	$2-(1)$	$3-(1)$	$2-(2)$
$=0$	$=1$	$=2$	$=0$

$R_3 \to R_3 - 2R_1$

$2-2(1)$	$1-2(1)$	$-1-2(1)$	$3-2(2)$
$=0$	$=-1$	$=-3$	$=-1$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -3 & -1 \end{array}\right]$

Step 3: Make every entry below the leading 1 in Column 2 zero

$R_3 \rightarrow R_3 + R_2$

$0 + (0)$	$-1 + (1)$	$-3 + (2)$	$-1 + (0)$
$= 0$	$= 0$	$= -1$	$= -1$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$

Step 4: Make the leading entry of Row 3 equal to 1

$R_3 \rightarrow R_3 \div (-1)$

$0 \div (-1)$	$0 \div (-1)$	$-1 \div (-1)$	$-1 \div (-1)$
$= 0$	$= 0$	$= 1$	$= 1$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Step 5: Make every entry above the leading 1 in Column 3 zero

$R_1 \rightarrow R_1 - R_3$

$1 - (0)$	$1 - (0)$	$1 - (1)$	$2 - (1)$
$= 1$	$= 1$	$= 0$	$= 1$

$R_2 \rightarrow R_2 - 2R_3$

$0 - 2(0)$	$1 - 2(0)$	$2 - 2(1)$	$0 - 2(1)$
$= 0$	$= 1$	$= 0$	$= -2$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Step 6: Make every entry above the leading 1 in Column 2 zero

$R_1 \rightarrow R_1 - R_2$

$1 - (0)$	$1 - (1)$	$0 - (0)$	$1 - (-2)$
$= 1$	$= 0$	$= 0$	$= 3$

NEW MATRIX $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

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Step 7: Read off the answers

Each row now says one variable equals one number.

$x = 3$

$y = -2$

$z = 1$

ANSWER

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & -2 \\ 0 & 0 & \textcircled{1} & 1 \end{array}\right]$$

$\boldsymbol{x = 3,\ y = -2,\ z = 1}$

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